Experimental search for the effect of compressibility in unsteady Couette flow

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Solution of the compressible Rayleigh problem involving an impulsively moved flat plate indicates that transverse velocities can be generated by a purely longitudinal shearing motion. It would be desirable to demonstrate the existence of these waves, and to do so was the motivation for the present investigation. However, the geometry for the Rayleigh problem is prohibitive; and since it was noticed that a similar behaviour should occur for the case of unsteady compressible Couette flow, this was considered for the present investigation. An experimental search for the existence of the transverse velocities was conducted using concentric rotating cylinders to generate the flow. Since these transverse velocities were too small to be measured directly, a temperature gradient was created across the annulus to take advantage of the much greater sensitivity of a hot-wire to temperature fluctuations produced by the fluctuation in the transverse velocity. The periodic temperature fluctuations noted were converted to equivalent transverse velocities. Experimental values obtained agreed qualitatively with the theory. Finally, the role of extraneous effects in the experimental apparatus was considered.

1. Introduction

Relation to the Rayleigh problem

The results of the classical incompressible Rayleigh problem, an infinite plate moved impulsively from rest parallel to its own plane, indicate that the resulting fluid motion is everywhere parallel to the plate and can be expressed rather simply in terms of the error function. Recently this problem has been re-examined for compressible fluids by a number of authors. In this case the conservation equations are coupled, basically through the viscous dissipation terms, and a non-zero transverse velocity occurs. In fact, as Stewartson (1955) shows, a strong shock wave travelling away from the plate can be generated. The total flow field is made up of three regions: a viscous boundary layer close to the moving plate, a disturbed inviscid region between the boundary layer and shock, and an undisturbed region beyond the shock wave. Mathematically the problem is to obtain separate solutions in these regions, and then to match them at the boundaries.

Illingworth (1950) has given a boundary layer solution suitable for large time and arbitrary Mach number. Howarth (1951) was able to obtain an expression for the pressure field in the compressible case by a linearization process, and he also examined the initial motion without recourse to linearization. Van Dyke (1952) treated the non-linear problem and obtained a solution valid for small Mach numbers. He obtained a solution for the boundary layer region and the outer flow field to the third approximation by iterating between the two solutions. His process could be repeated indefinitely, except that the shock discontinuity in the outer flow field would have to be included. Finally, Stewartson (1955) considered the flow field for large Mach numbers, on the assumptions that the outer inviscid flow field and the boundary layer could be treated separately. A detailed summary of the existing theories is given by Stewartson (1955).

Van Dyke's and Stewartson's solutions together cover the complete Mach number range for the impulsive plate problem. Stewartson (1955) has compared the two solutions and found that even though there is a discontinuity, the agreement is fairly close, and that they can be matched by an approximate solution.

In each of the papers on the Rayleigh problem the distinguishing feature between the incompressible and compressible case is emphasized. This, of course, is the presence of the outflow velocity in the compressible case arising from the fact that viscous dissipation heats the fluid and a corresponding expansion occurs. Typical values of this outflow velocity $\bar{v}$ can be computed from the analyses mentioned. For example, Van Dyke (1952) gives the following expression for the small Mach number range in terms of the Mach number, time, and fluid properties:

$$\bar{v} = \frac{\gamma - 1}{\sqrt{(2\pi)}} M^2 \sqrt{\left(\frac{\nu}{t}\right)}, \tag{1}$$

where γ is the ratio of the specific heats and ν the kinematic viscosity. Assuming first, as an upper linit, $M = 1$ and t arbitrary—say $\frac{1}{4}$ sec, equation (1) gives for air $\bar{v} = 4 \times 10^{-3}$ ft./sec. For a low speed $M = 0{\cdot}01$, equation (1) gives

$$\bar{v} = 4 \times 10^{-7}\,\text{ft./sec.}$$

It should be noticed that these are extremely small values. However, the overall results of the compressible Rayleigh problem indicate that transverse waves, or velocities, can be generated by a pure longitudinal shearing motion. It would be desirable to demonstrate the existence of these transverse waves experimentally. This was the motivation for the present investigation.

Experimental verification with a very large impulsive movement of a plate would be difficult, and so the question was raised as to whether the same phenomenon could be demonstrated with a different geometry, although in an analogous fashion. Examination of the basic equations for compressible plane Couette flow reveal the same degree of coupling as for the impulsive plate, i.e. transverse waves should occur in the channel for an unsteady motion of the moving walls. Also, for Couette flow the wall motion need not be impulsive, but could be a small oscillation superimposed on a uniform mean motion.

This reasoning led to the use of concentric rotating cylinders (outer cylinder rotating) to assimilate the plane Couette motion. It is realized that steady flow between concentric cylinders does not result in a linear velocity profile, but the deviation from linearity is small, i.e. 3 to 4 %, with the geometry which was adopted. The oscillation of the outer wall is primarily felt in a small region close to it, and so the approximation with circular cylinders was found to be quite satisfactory.

With the circular geometry the problem was then to determine whether any periodic transverse velocity existed which could be traced to the wall velocity perturbation.

2. Analytical discussion of the Couette problem

Unsteady incompressible solution

So far it has not been possible to obtain a solution for unsteady compressible plane Couette flow which is suitable for comparison with experimental results. For the incompressible case, DeGroff (1955) has given a solution, and because of its significance in the present investigation it will be reviewed here. The equations of motion reduce to the diffusion equation (as for the incompressible Rayleigh problem)

$$\frac{\partial \bar{u}}{\partial t} = \nu \frac{\partial^2 \bar{u}}{\partial y^2}, \tag{2}$$

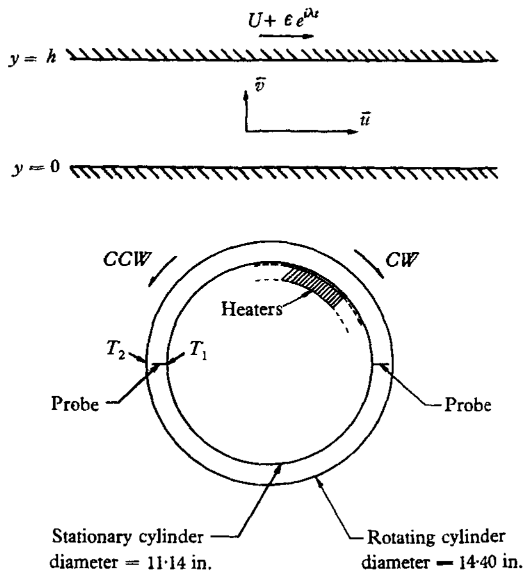

FIGURE 1. Notation for unsteady plane Couette flow and orientation of circular cylinders.

with the boundary conditions $\bar{u}(0,t) = 0$ and $\bar{u}(h,t) = U + \epsilon e^{i\lambda t}$. The co-ordinate system and notation are as shown in figure 1. In this case, as for the Rayleigh problem, conservation of mass for incompressible fluid motion precludes the existence of a transverse velocity.

The real part of the solution to (2) with the boundary conditions applied is given by

$$\bar{u}(y,t) = \frac{U}{h}y + \frac{\epsilon}{4A}\{e^{\alpha(h+y)}\cos(\lambda t + \alpha y - \alpha h) - e^{-\alpha(h-y)}\cos(\lambda t + \alpha y + \alpha h) - e^{\alpha(h-y)}\cos(\lambda t - \alpha y - \alpha h) + e^{-\alpha(h+y)}\cos(\lambda t - \alpha y + \alpha h)\}, \tag{3}$$

where $A = \sinh^2 \alpha h \cos^2 \alpha h + \cosh^2 \alpha h \sin^2 \alpha h$ and $\alpha = \surd(\lambda/2\nu)$. The first part of the solution is the linear steady-state Couette profile, and the second part is the contribution from the real part of the perturbation on the wall. It is interesting to note that the solution is made up of four components at different phase angles. Since $0 \leqslant y \leqslant h$ and generally $\alpha \gg 1$, it can be seen that in the outer portion of the channel the first term in the brackets is by far predominant over the other three.

Analytical determination of the transverse velocity

The compressible case differs from the incompressible in that the conservation equations are highly coupled and the full set of five must be solved simultaneously. One of the means of coupling is afforded through viscous dissipation, with the result that the velocity plays an important role in the energy equation. Also (2), which expresses conservation of momentum in the direction of $\overline{u}$, now involves both velocities $\overline{u}$ and $\overline{v}$ and the density $\overline{\rho}$. Thus,

$$\overline{\rho}\overline{u}_t + \overline{\rho}\overline{v}\overline{u}_y = \mu\overline{u}_{yy}, \tag{4}$$

where the subscripts denote partial differentiations.

Linearization of the equations is one means of attack used by Howarth (1951) for the flat plate, and if this is done here a particular characteristic of the present problem is seen. The linearization here is performed on the basis of a perturbation superimposed on the steady incompressible motion. The linearizing relations are $\overline{\rho} = \rho_0 + \rho$, $\overline{v} = v$ and $\overline{u} = (Uy/h) + u$; hence, if squares of the perturbation quantities ρ, v and u are neglected, there results from (4)

$$u_t + \frac{U}{h} v = \frac{\mu}{\rho_0} u_{yy}. \tag{5}$$

It is to be noted that the transverse velocity is retained in this equation, and thus the linearized equations are still coupled. This is in contrast with the linearized Rayleigh problem which depends on viscous dissipation for the coupling. It suggests that viscous dissipation can be completely neglected while still allowing an examination of coupling effects. This was done in the present case, with the viscous dissipation being made small experimentally due to the low air-flow velocities considered.

The coupling in the Rayleigh problem depends primarily on a thermodynamic mechanism, whereas the Couette case is seen to exhibit a strong dynamical mechanism. From the left-hand side of (5) it can be seen that for an inviscid fluid a transverse velocity must exist to satisfy the fluid acceleration in unsteady shear flows ($u_t \neq 0$, $u_y \neq 0$).

The linearization of (4) indicates that u_t and Uv/h should be of the same order of magnitude. A complete solution confirming this is not available; however, it is possible to compare the two terms by the following approximate analysis.

Even though there is no boundary layer in the usual sense, the region near the upper wall which is primarily affected by the perturbation can be considered as one, and from this a suitable 'boundary layer displacement thickness' defined. As an order of magnitude approximation this displacement thickness can be found using the incompressible solution; and since it is a time-dependent function, its time rate of change is indicative of the transverse velocity.

Following this procedure, the displacement thickness δ^* is defined as

$$\delta^* = \int_0^h \frac{u}{\epsilon} dy. \tag{6}$$

Obtaining u from (3) and integrating with respect to y results in

$$\delta^* = \frac{1}{8A\alpha}\{2\sinh 2\alpha h(\cos\lambda t+\sin\lambda t) - 2\,e^{\alpha h}[\cos(\lambda t-\alpha h)+\sin(\lambda t-\alpha h)] + 2\,e^{-\alpha h}[\cos(\lambda t+\alpha h)+\sin(\lambda t+\alpha h)] + \sin 2\alpha h(-\cos\lambda t+\sin\lambda t)\}. \tag{7}$$

The first term will, in general, be much larger than the last three, so that

$$\delta^* = \frac{1}{2\alpha}(\cos\lambda t+\sin\lambda t) = (\nu/2\lambda)^{\frac{1}{2}}(\cos\lambda t+\sin\lambda t). \tag{8}$$

The value of δ^* for a flat plate oscillating alone in an infinite fluid is given by $\delta^* = \sqrt{(2\nu/\lambda)}$, which is $\sqrt{2}$ times the maximum value for the present problem. This is reasonable, since it would be expected that the present thickness would be smaller due to the proximity of the fixed plate.

Next, since $v = d\delta^*/dt$, differentiation of (7) gives

$$v = \frac{\lambda}{4A\alpha}\{\sinh 2\alpha h(\cos\lambda t-\sin\lambda t) - e^{\alpha h}[\cos(\lambda t-\alpha h)-\sin(\lambda t-\alpha h)] + e^{-\alpha h}[\cos(\lambda t+\alpha h)-\sin(\lambda t+\alpha h)] + \sin 2\alpha h(\cos\lambda t+\sin\lambda t)\}. \tag{9}$$

Again the first term is dominant near the outer wall; and if the time is chosen so that v is maximum, then

$$v_{\max} = \frac{\lambda}{\sqrt{2}\,\alpha} = \sqrt{(\lambda\nu)}. \tag{10}$$

Returning to (3), the value of u_t can be computed. At the wall this will simply be $\epsilon\lambda\sin\lambda t$ from the boundary condition. The order of magnitude for the ratio of the first two terms in (5) is then given by

$$\frac{u_t}{Uv/h} = O\left[\frac{h\epsilon}{U}\sqrt{(\lambda/\nu)}\right] = O\left[\frac{\epsilon}{U}\frac{h}{\delta^*}\right]. \tag{11}$$

Evaluated at the wall, where u_t is a maximum, (11) gives a ratio of $(u_t h)/(Uv) = 2$ for the experimental range of variables used here. If the ratio is to be found in the region where the experimental measurements were made, u_t must be computed using (3) as indicated above, evaluating the derivative at the co-ordinate desired. If $y = 0{\cdot}9h$ is arbitrarily chosen, $(u_t h)/(Uv) = 0{\cdot}5$. The transverse velocity as obtained here is independent of the co-ordinate y. However, the result should give the right order of magnitude. The results indicate that even at the wall, or at maximum u_t, the two terms are of the same order of magnitude; and thus the coupling due to the shear field is important and cannot be neglected.

The quantity h does not have any significance in the impulsive plate problem, but if the transverse velocity v is arbitrarily multiplied by a scale factor U/h, then

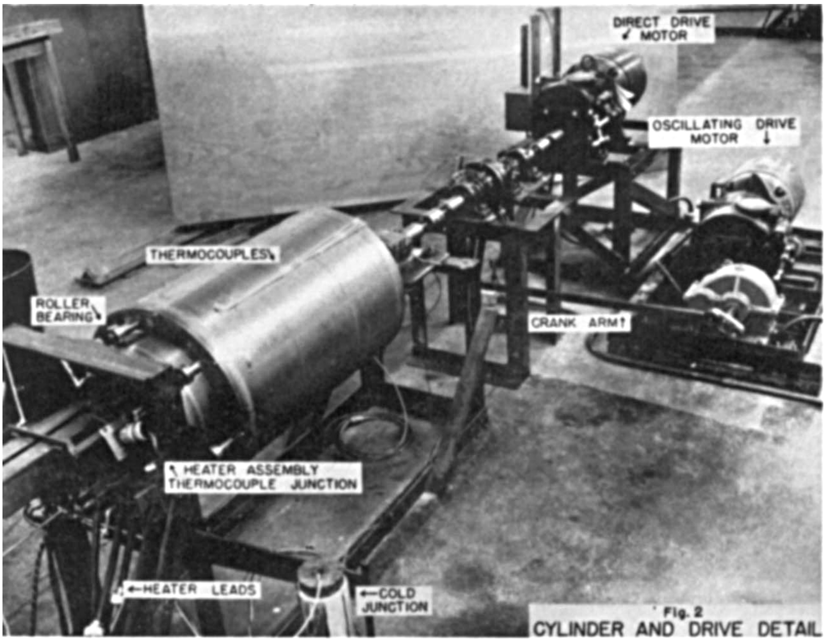

FIGURE 2 (plate 1). View of cylinder and drive.

for Van Dyke's solution (1) we have $(u_t h)/(Uv) \sim 10^6$ for the range of variables used above. This is much larger than for the present case. Since u_t is the same order of magnitude for the two cases, a much larger v should be obtained experimentally in the present case than the 4×10^{-7} ft./sec indicated by (1).

3. Experimental verification

Apparatus

The two metal cylinders used were 11·14 in. and 14·40 in. in diameter and were placed with their axes horizontal. The resulting annulus was 1·63 in. across and 22 in. in axial length. Profile velocity and temperature measurements were made in the horizontal plane at the axial mid-point of the annulus. The geometry and co-ordinate system for the cylinders are shown in figure 1. A photograph of the apparatus along with the driving mechanisms is shown in figure 2, plate 1. The report by Hromas & Thompson (1956), which deals with the steady-state heat transfer between the cylinders, gives a complete description of the apparatus.

All the measurements were made with hot-wire anemometers used as resistance thermometers for temperature measurements and in the normal manner for velocity measurements. The development and calibration of the probes has been described in the report by Hromas & Kentzer (1955).

Experimental technique

The objective was to examine the unsteady motion in the annulus, and, in particular, to determine whether any transverse velocity existed due to the imposed perturbation. At the outset one primary factor concerning the phenomenon was realized. This was that any existing transverse velocity would be very small, and hence any deviation from an incompressible behaviour would be slight. Therefore, the longitudinal velocity profile data should agree fairly well with the relatively simple incompressible solution.

With this fact in mind, the first step was to obtain a series of longitudinal velocity profile measurements for several shaking frequencies, and to compare them with the incompressible analysis.

The results indicated that any transverse velocity was indeed small, which led to the realization that it would not be suitable to attempt to measure the transverse velocity component directly. It was then proposed that it might instead be possible to measure v indirectly through its effect on another variable. In particular, it was thought that, since temperature changes could be detected on a much smaller scale than velocity, temperature fluctuations might be detected due to the oscillating wall. Using the hot-wire as a resistance thermometer, it was conjectured that if a temperature gradient were to be placed across the flow, any transverse velocity component would have a magnified effect on the temperature. From data obtained during the previous steady-state heat-transfer experiments of Hromas & Thompson (1956), suitable gradients were chosen such that the effect of free convection was known to be negligible.

When a temperature gradient was placed across the flow, definite temperature fluctuations were noted to occur at the frequency of the perturbation oscillation of the rotating cylinder. In order to extend the range of data, several different temperature gradients were used with temperature traces being recorded at various gap settings in the annulus for each gradient. Instrumentation response determined the lowest gradient usable, and free convection the largest.

It was concluded that by this indirect method the existence of a transverse velocity was demonstrated for the problem involved. Assuming that $v\Delta T$ is indicative of the forced convection, and then knowing the temperature gradient, the frequency of oscillation, and the magnitude of the temperature fluctuations, average values for the transverse velocity were computed.

Comparison between theory and experiment

The first data obtained were the resultant velocity traces for various frequencies and gap settings. For both of these the cylinder walls were at room temperature. The experimental data are compared in figure 3 with the theoretical results for unsteady incompressible plane Couette flow. It can be seen that the experimental data agreed very closely with the analysis. The maximum discrepancy corresponds to a velocity of 0·07 ft./sec; and as the sensitivity limit of the instrumentation was 0·05–0·07 ft./sec, this error is not significant.

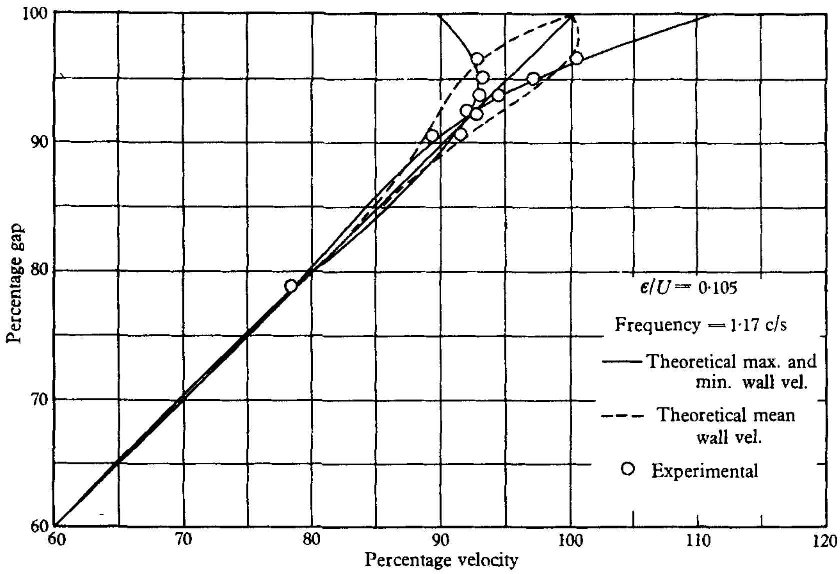

FIGURE 3. Theoretical and experimental velocity profiles for unsteady incompressible Couette flow.

Further evidence of the close check with the incompressible analysis can be seen in figure 4. These were data obtained for a given gap setting and various frequencies, keeping constant the ratio ϵ/U of the perturbation magnitude to the mean wall velocity. The four data points fall about 0·1 % below the theoretical values, corresponding to velocity differences of only 0·007 to 0·010 ft./sec. Within the accuracy of the instrumentation the above results certainly indicate a close check with the incompressible theory. Further, they confirm the initial assumption that any transverse velocity would be small.

When a temperature gradient was formed across the flow, oscillations were noticed. Figure 5 gives the magnitude of these oscillations as a function of gap setting and temperature gradient. The large degree of scatter was primarily caused by difficulties of measurement. Also, the values given in figure 5 were measured peak to peak so the maximum deviation from the mean was about 1·5° F.

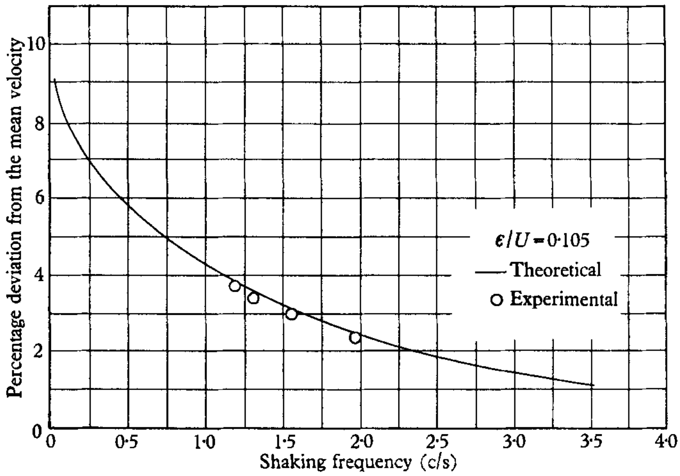

FIGURE 4. Instantaneous velocity fluctuation *vs* shaking frequency at 96·5 % gap.

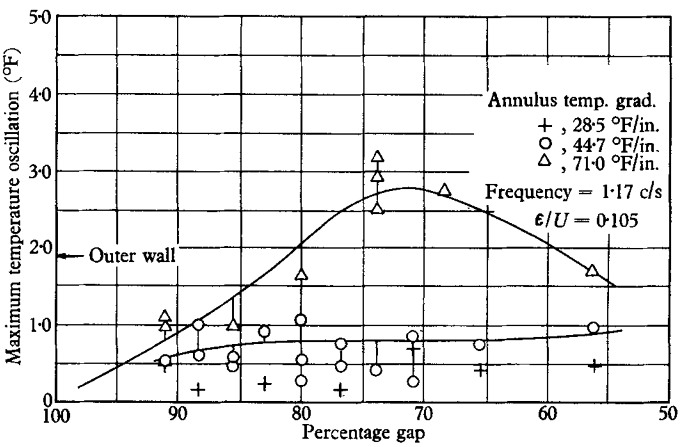

FIGURE 5. Maximum temperature oscillation *vs* percentage gap for various annulus temperature gradients.

The computed values of the transverse velocity are shown in figure 6. Values obtained range from about 0·002 ft./sec to 0·008 ft./sec. Evidently, such small values could not have been measured directly with the hot-wire.

Since numerical values for v are now obtained, a comparison can be made with the tentative theoretical values obtained from the computed displacement thickness and the linearized compressible equation. Previously, the ratio of $(u_t h)/(Uv)$ at $y = 0{\cdot}9h$ had been determined theoretically and found to be 0·5.

If now u_t is computed from the experimental data and v taken to be about 0·003 ft./sec, there results

$$\left[\frac{u_t h}{Uv}\right]_{\text{exp}} = 1{\cdot}5,$$

which indicates at least an order of magnitude agreement. In view of the nature of the experiments, this is really all that could be expected.

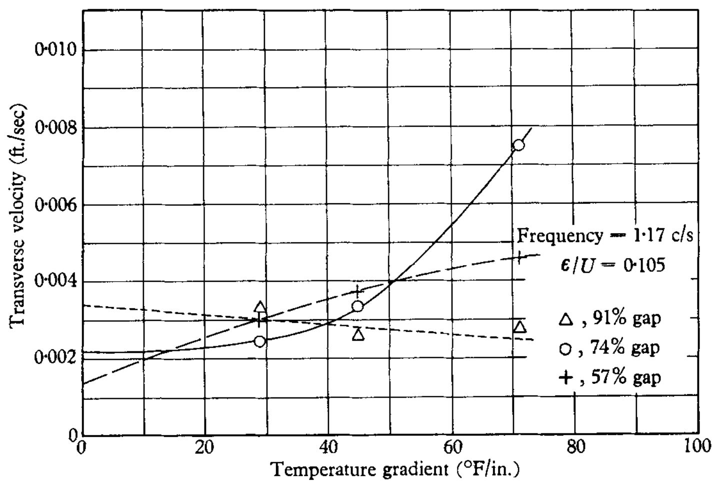

FIGURE 6. Transverse velocity component *vs* annulus temperature gradient.

4. Discussion of results

Temperature fluctuations

From figure 5 it can be seen that the magnitude of the temperature fluctuations increases with the temperature gradient. In fact, it was found to be roughly proportional to the magnitude of the gradient. It would be desirable to check this further, but the instrumentation and free convection set rather definite upper and lower limits on the gradients usable.

The high temperature-gradient readings appeared to have a peak at about the 70 % gap position. Why this occurred is not clear. There could possibly have been some sort of resonance effect present. A careful examination of the other two sets of data shows that the first one or two points tend to be high, so they might also have been drawn with a slight peaking effect close to the outer wall. Without the aid of an analytical solution it would be difficult to suggest anything of a more definite character.

Transverse velocity fluctuations

Since the presence of a transverse component of velocity was detected only with a temperature gradient, a question of prime importance was whether the gradient was a necessary condition for the existence of transverse velocities. Since the gradients used were about the upper and lower limits, the only recourse was to resort to an extrapolation to zero temperature gradient with the data available. This was done and the resulting curves are shown on figure 6. This procedure was not precise, but it appears from all the data that a non-zero transverse velocity would be obtained at room temperature. The values indicated are of the order of 0·003 ft./sec.

The extrapolation did not change the order of magnitude of the observed velocity, and hence gives a result which still agrees with the analysis. For this reason it was felt the extrapolation was justified. Thus it was concluded that, as indicated to be true by the coupling in the basic equations, a periodic transverse velocity would exist at zero temperature gradient.

Possibility of extraneous effects

Four sources of error which would have affected the results are free convection, centrifugal effects, instability regions in the annulus (even though the outer cylinder was rotated) and viscous dissipation. For the velocities used one would expect viscous dissipation to be very small. Using the results of DeGroff (1956),

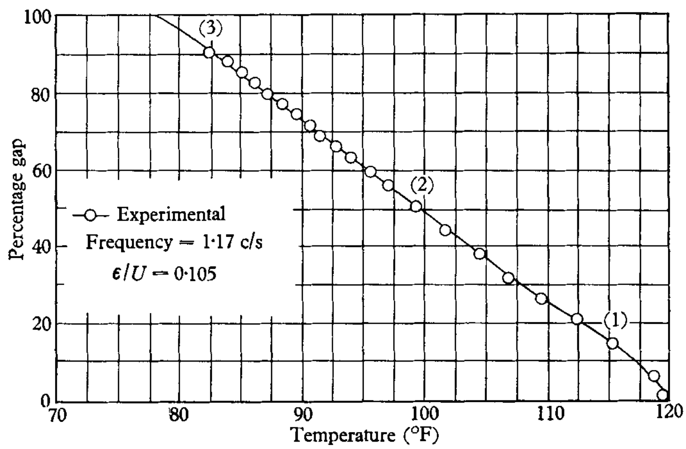

FIGURE 7. Mean temperature profile for unsteady flow (annulus temperature gradient—28·5 °F/in.).

the dissipation was found in this case to raise the fluid temperature a maximum of 0·0007 °F. This is certainly not detectable in the temperature fluctuations. Concerning an instability region, it is felt that the extensive hot-wire measurements which had been made in the previous steady-state investigations would have indicated any that existed.

It was found in the steady-state heat-transfer measurements that free convection had a large effect near the inner cylinder and in a small confined layer along the outer cylinder. These two regions were connected by a linear portion free from convection currents. Figure 7 shows a profile for a particular case chosen to minimize free convection effects. For all gradients used in the present work, the data obtained were restricted to well within the linear regions and hence should be independent of free convection.

Finally, there is the possibility that the transverse velocities noticed could have been caused by centrifugal effects. Continuity considerations require that regardless of whether the motion is cylindrical or plane, the existence of a transverse velocity for the present problem requires some degree of compressibility in the fluid. However, centrifugal force can still contribute to the magnitude of the transverse velocity, and it remains to see how this compares with the effects of the basic coupling as indicated in the equations for plane flow. The coupling for plane compressible Couette flow was indicated by the linearized momentum equation (5). The corresponding equation for unsteady compressible cylindrical Couette flow will now be linearized on the same basis as the plane case, i.e. a small perturbation is superimposed on the steady incompressible motion. The linearization relations are $\bar{\rho} = \rho_0 + \rho$, $\bar{q}_\theta = q_{\theta_0} + q_\theta$ and $\bar{q}_r = q_r$, where the subscript 0 corresponds to the steady incompressible state. The resulting equation for the perturbation velocities is

$$\rho_0\left[\frac{\partial q_\theta}{\partial t} + q_r\left(\frac{\partial q_{\theta_0}}{\partial r} + \frac{q_{\theta_0}}{r}\right)\right] = \mu\left[\frac{\partial^2 q_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r^2}\right]. \tag{12}$$

The second term on the left-hand side is analogous to the coupling term in (5), and the last term on the left-hand side occurs due to the cylindrical nature of the flow. However, the relative magnitude of these two terms can be readily determined from the relation for the steady incompressible profile

$$q_{\theta_0} = \frac{\omega r_2^2}{r_2^2 - r_1^2}\left[r - \frac{r_1^2}{r}\right]. \tag{13}$$

The ratio between the second and third terms is then

$$\frac{\partial q_{\theta_0}/\partial r}{q_{\theta_0}/r} = \frac{1 + (r_1^2/r^2)}{1 - (r_1^2/r^2)}, \tag{14}$$

which for the present apparatus takes the value 4 at $r = r_2$, and approaches infinity as $r \to r_1$. This indicates that centrifugal forces are felt in the experimental apparatus, but are never predominant, being 25 % of the effect at the outer wall and approaching zero at the inner wall. For the region of the annulus in which the data were obtained, centrifugal force could have caused about 10 to 15 % of the measured effect.

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